Practical Aspects for Implementation of Fractional-Order Controllers

Ivo Petráš
Faculty of BERG
Technical University of Košice
B. Němcovej 3, 042 00 Košice, Slovak Republic
Email: ivo.petras@tuke.sk

***Abstract*—This paper deals with practical aspects of fractional-order controller implementation. Technique for digital implementation of fractional operators are presented as well as some practical notes to implementation of fractional integral and fractional derivative. Elimination of wind-up effect and limitation of derivative action are presented in order to implement such kind of controller in practice.**

Keywords–fractional calculus, fractional-order controller, wind-up effect, saturation, filtering derivative action.

I. INTRODUCTION

Fractional calculus is known since the regular calculus with the first written reference dated in September 1695. This mathematical phenomenon is connected with the famous names as for example L'Hospital, Euler, Fourier, Leibniz, Riemann, Liouville, etc. Nowadays, the fractional calculus has a wide area of applications, for instance bioengineering [8], physics [6], chaos theory [21], viscoelasticity [9], and many others (see e.g. [13], [24]). One of another possible application is the control system engineering. During the last 20 years a huge effort has been made to describe various possibilities of how to implement the fractional calculus techniques in control theory [3], [11]. We can mention for example: new type of fractional-order controllers, new fractional-order model for the plant (process), etc. In this article we will focus on fractional-order controllers because of a wide area of applications. As already has been noted in [4], [5], fractional-order control, namely fractional PID controllers, could be ubiquitous in industry. For example a typical mill in Canada has more than 2000 control loops, where 97% loops are based on PI control. However, only 20% of control loops work well [1]. The reason is bad tuning, incorrect implementation techniques, some restrictions and limitations, actuator and sensor problems, and so on.

For elimination of above problems we can use a several modification usually used in standard control techniques implementation, as for instance, wind-up effect elimination, limitation and filtering a derivative action, setpoint tracking.

This paper is organized as follows: In Section I is briefly discussed some introduction to problem. Section II is focused on fractional calculus fundamentals. In Section III are described some discrete implementation techniques for fractional operators. Section IV brings a survey of some control actions limitations and modifications. In Section V is described an illustrative example. Section VI concludes this paper with some additional remarks and idea for further work.

II. FRACTIONAL CALCULUS FUNDAMENTALS

A. Fractional-order integro-differential operator

Fractional calculus is a generalization of integration and differentiation to non-integer order fundamental operator ${}_aD_b^\alpha$, where a and b are the bounds of the operation. The standard notation for denoting the left-sided fractional-order integro-differential operator of a function $f(t)$ defined in the interval $[a, b]$ is ${}_aD_t^\alpha f(t)$, with $\alpha \in \mathrm{R}$.

There exist three main definitions of the fractional order integrals and derivatives: Riemann-Liouville, Caputo, and Grünwald-Letnikov [22].

Definition 1: Riemann-Liouville definition of fractional derivative is given as [12], [22]:

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \qquad (1)$$

for $(n-1 < \alpha < n)$, where $\Gamma(\cdot)$ is Euler's *Gamma* function.

Definition 2: Caputo's definition of fractional derivatives can be written as [22]:

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \qquad (2)$$

for $(n-1 < \alpha < n)$. The initial conditions for fractional-order differential equations with Caputo's derivative are in the same form as for integer-order differential equations.

Definition 3: If we consider $k = \frac{t-a}{h}$, where a is a real constant, which expresses a limit value, we can write the Grünwald-Letnikov definition as [12], [22]:

$$ {}_aD_t^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{j=0}^{[k]} (-1)^j \binom{\alpha}{j} f(t-jh), \qquad (3)$$

where $[x]$ means the integer part of x, a and t are the bounds of operation for ${}_aD_t^\alpha f(t)$. This form of definition is very helpful for obtaining a numerical solution of fractional differential equations.

For binomial coefficients calculation we can use the relation between Euler's *Gamma* function and factorial, defined as follows

$$ \binom{\alpha}{j} = \frac{\alpha!}{j!(\alpha-j)!} = \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\,\Gamma(\alpha-j+1)}. \qquad (4)$$

978-1-4799-3528-4/14/$31.00 ©2014 IEEE

For zero initial conditions and lower terminal $a = 0$, the Laplace transform of fractional derivatives (Riemann-Liouville, Grünwald-Letnikov, and Caputo), reduces to [22]:

$$\mathscr{L}\{{}_0D_t^{\alpha} f(t)\} = s^{\alpha} F(s). \qquad (5)$$

Some other integral transformations and additional important properties of the fractional derivatives and integrals can be found in several works (e.g. [8], [11], [12], [13], [22]).

B. Fractional-order systems

The control systems can include both the fractional order dynamic system to be controlled and the fractional-order controller. A fractional order plant can be described by a typical n-term linear fractional-order differential equation (FODE) in time domain:

$$a_n\, D_t^{\beta_n} y(t) + \cdots + a_1\, D_t^{\beta_1} y(t) + a_0\, D_t^{\beta_0} y(t) = 0, \qquad (6)$$

where $a_k (k = 0, 1, \cdots, n)$ are constant coefficients of the FODE; $\beta_k, (k = 0, 1, 2, \cdots, n)$ are real numbers. Without loss of generality, assume that $\beta_n > \beta_{n-1} > \cdots > \beta_1 > \beta_0 \geq 0$. Consider a control function which acts on the FODE system (6) as follows:

$$a_n\, D_t^{\beta_n} y(t) + \cdots + a_1\, D_t^{\beta_1} y(t) + a_0\, D_t^{\beta_0} y(t) = u(t). \qquad (7)$$

By Laplace transform technique, we can get a fractional transfer function:

$$G_p(s) = \frac{Y(s)}{U(s)} = \frac{1}{a_n s^{\beta_n} + \cdots + a_1 s^{\beta_1} + a_0 s^{\beta_0}}. \qquad (8)$$

In general, a fractional-order dynamic system can be represented by a transfer function of the form:

$$G_p(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^{\alpha_m} + \cdots + b_1 s^{\alpha_1} + b_0 s^{\alpha_0}}{a_n s^{\beta_n} + \cdots + a_1 s^{\beta_1} + a_0 s^{\beta_0}}, \qquad (9)$$

where $a_k\ (k = 0, \ldots n)$, $b_k\ (k = 0, \ldots m)$ are constant, and $\alpha_k\ (k = 0, \ldots n)$, $\beta_k\ (k = 0, \ldots m)$ are arbitrary real or rational numbers and without loss of generality they can be arranged as $\alpha_n > \ldots > \alpha_1 > \alpha_0$, and $\beta_m > \ldots > \beta_1 > \beta_0$.

C. Fractional-order controllers

The fractional-order PI$^{\lambda}$D$^{\delta}$ controller (FOC) was proposed in [23] as a generalization of the PID controller with integrator of real order λ and differentiator of real order δ. The transfer function of such controller in the Laplace domain has the form:

$$C(s) = \frac{U(s)}{E(s)} = K_p + T_i\, s^{-\lambda} + T_d\, s^{\delta}, \qquad (\lambda, \delta > 0), \ (10)$$

where K_p is the proportional constant, T_i is the integration constant and T_d is the differentiation constant.

The internal structure of the fractional-order controller consists of the parallel connection, the proportional, integration, and derivative part. The transfer function (10) corresponds in time domain to the fractional differential equation of the form:

$$u(t) = K_p\, e(t) + T_i\, {}_0D_t^{-\lambda} e(t) + T_d\, {}_0D_t^{\delta} e(t), \qquad (11)$$

or discrete transfer function given in the following expression:

$$C(z) = \frac{U(z)}{E(z)} = K_p + \frac{T_i}{(\omega(z^{-1}))^{\lambda}} + T_d(\omega(z^{-1}))^{\delta}, \qquad (12)$$

where $\omega(z^{-1})$ denotes the discrete operator, expressed as a function of the complex variable z or the shift operator z^{-1}.

Taking $\lambda = 1$ and $\delta = 1$, we obtain a classical PID controller. If $\lambda = 0$ and $T_i = 0$, we obtain a PD$^{\delta}$ controller, etc. All these types of controllers are particular cases of the fractional-order controller, which is more flexible and gives an opportunity to better adjust the dynamical properties of the fractional-order control system.

It can also be mentioned that there are many other modification of the fractional PI$^{\lambda}$D$^{\delta}$ controller [7], [10], [11], [16], [26] and another considerations of the fractional controllers.

III. Digital implementation techniques

Implementation techniques for the FOC have been described in several works. Some proposals we can be found in work [25]. An analogue implementation was proposed in book [19] and a digital implementation was suggested in works [3], [15]. In this paper we will focus only on the digital implementation techniques. Having tuned the controllers, to implement them we have to take into account other considerations, such as memory size and computational load required by the algorithm, knowing that, in any case, the fractional orders must be approximated by integer ones. Implementation of the FOC in Matlab as a function by using three different techniques was done in [17].

In general, if a function $f(t)$ is approximated by a grid function, $f(kh)$, where h is the grid size, the approximation for its fractional derivative of order r can be expressed as [3], [11]:

$$y_h(kh) = h^{\mp r} \left(\omega\left(z^{-1}\right)\right)^{\pm r} f_h\left(kh\right), \qquad (13)$$

where z^{-1} is the backward shift operator and $\omega\left(z^{-1}\right)$ is a generating function. This generating function and its expansion determine both the form of the approximation and the coefficients. In this way, the discretization of continuous fractional-order differentiator/integrator $s^{\pm r}$ $(r \in \mathrm{R})$ can be expressed as $s^{\pm r} \approx (\omega(z^{-1}))^{\pm r}$.

As a generating function $\omega(z^{-1})$ the following formula can be used in general [2]:

$$\omega(z^{-1}) = \left(\frac{1}{\beta T} \frac{1 - z^{-1}}{\gamma + (1-\gamma) z^{-1}}\right), \qquad (14)$$

where β and γ are denoted the gain and phase tuning parameters, respectively, and T is the sampling period. For example, when $\beta = 1$ and $\gamma = \{0, 1/2, 7/8, 1, 3/2\}$, the generating function (14) becomes the forward Euler, the Tustin, the Al-Alaoui, the backward Euler, the implicit Adams rules, respectively. In this sense the generating formula can be tuned more precisely.

The expansion of the generating functions can be done by Power Series Expansion (PSE) or Continued Fraction Expansion (CFE). It is very important to note that PSE scheme leads to approximations in the form of polynomials of degree p, that is, the discretized fractional-order derivative is in the form of FIR filters, which have only zeros. The CFE scheme leads to approximations in the form of rational function and the discretized fractional-order derivative is in the form of IIR filters.

Then, the resulting transfer function, approximating the fractional-order operators via PSE method, can be obtained by applying the relationship

$$D^{\pm r}(z) = \frac{Y(z)}{F(z)} \approx \ \text{PSE}\left\{(\omega(z^{-1})^{\pm r}\right\}_p \simeq P_p(z^{-1}), \quad (15)$$

where $Y(z)$ is the Z transform of the output sequence $y(kT)$, $F(z)$ is the Z transform of the input sequence $f(kT)$, and PSE$\{u\}$ denotes the expression, which results from the power series expansion of the function u, $D^{\pm r}(z)$ denotes the discrete equivalent of the fractional-order operator, considered as processes, and $P_p(z^{-1})$ is the polynomial with degree p of variable z^{-1}.

The resulting discrete transfer function, approximating fractional-order operators via CFE method, can be expressed as:

$$\begin{aligned} D^{\pm r}(z^{-1}) &= \frac{Y(z)}{F(z)} \approx \text{CFE}\left\{(\omega(z^{-1}))^{\pm r}\right\}_{p,q} \\ &\simeq \frac{P_p(z^{-1})}{Q_q(z^{-1})} = \frac{p_0 + p_1 z^{-1} + \ldots + p_m z^{-m}}{q_0 + q_1 z^{-1} + \ldots + q_n z^{-n}}, \end{aligned}$$

where CFE$\{u\}$ denotes the continued fraction expansion of u; p and q are the orders of the approximation and P and Q are polynomials of degrees p and q. Normally, we can set $p = q = n$.

IV. Modification of the fractional-order control actions

Several modifications of the fractional-order control can be used in order to avoid such problems as wind-up effect, actuator saturation and derivative action limitation. Among most used ones belong the following [18], [20]:

- *Filtering the desired value $r(t)$*: filtering the desired value $r(t)$, known as setpoint tracking, by first or second order filter is a very frequently used trick to avoid problem with derivative action. Instead of step change of the desired value, which could be a problem especially for derivative part in controller, the control algorithm executes slow change of the desired value and changes of the control signal are not that extreme. For most applications, a first-order filter is satisfactory. We recommend the first-order prefilter in the form:

  $$H_p(z) = \frac{k_f}{1 - k_f z^{-1}},$$

  where k_f is the prefilter constant.

- *Using a controlled value in proportional and derivative parts of controller:* above-mentioned problem related to step changes of control signal due to step changes of desired value $r(t)$ can also be solved via replacing the control error $e(t) = r(t) - y(t)$ by controlled value $y(t)$. This modification can help a lot, especially, when desired value has changed rapidly and therefore the actuator becomes saturated.

- *Filtering the derivative action:* due to noisy signal on measured controlled value, the differentiation of noise can involve inappropriate changes of control signal. Derivative action is more sensitive to higher-frequency terms in the inputs. Because of this the derivative part in the controller can be filtered by first- and second-order high frequency filter. For the first-order filter in derivative part and with a genuine integral action, we can write the transfer function of the fractional-order controller in the form:

  $$C(s) = \frac{U(s)}{E(s)} = K_p + T_i \frac{s^{1-\lambda}}{s} + \frac{T_d\, s^{\delta}}{T_f s + 1}, \ (\lambda, \delta > 0), \quad (16)$$

  where $T_f = N/T_d$ is the filter constant. For $N = 0$, we obtain the usual FOC described by relation (10).

- *Limitation of integral action:* this limitation is also known as wind-up of the controller. It is due to the fact that actuator has also limitations and for instance if the actuator is at the end position and the control error is not zero, integral part of the controller rapidly grows and the controller calculates unreal value of the control signal and therefore the actuator stays at the end position until the sign of control error is changed. This problem is known as wind-up or integral saturation and it can be solved via limitation of integral part in the controller. Another possibility of how to avoid wind-up is to introduce limiters of the desired values so that the controller output will never reach the actuator bounds. There are also some other standard anti-wind-up schemes, which can be also used for the fractional-order control, as for instance:
  - conditional integration,
  - automatic reset configuration,
  - back-calculation.

Obviously there are many other modifications of the control algorithms, which help us implement the fractional-order controller in practice. For instance, we can mention initial conditions for a non-impact controller connection to control loop, analog and digital filtering of measured values, etc.

V. Example

Using a weighted desired value (setpoint) and limitation on control action in order to avoid actuator saturation and wind-up effect has been used and described in papers [18], [20]. Some standard anti-windup strategies together with proposed new one was described in work [14]. In this section an effect of a filtered derivative action is demonstrated.

Let us consider filtered derivative action in the FOC (10) as it was demonstrated in equation (16). Thus we obtain system:

$$D(s) = \frac{T_d\, s^{\delta}}{T_f s + 1}, \quad (\delta > 0), \quad (17)$$

where $T_f = N/T_d$ is the filter constant and $N \geq 0$. Following the well-know technique of the order compensation in classical PID controller, which is:

$$\ldots + T_2 u^{''}(t) + T_1 u^{'}(t) + u(t) = K_p e(t) + T_i \int e(t)dt + T_d e^{'}(t)$$

we can rewrite equation (17) to the following general form:

$$D(s) = \frac{T_d\, s^{\delta}}{T_f s^{\mu} + 1}, \quad (\delta, \mu > 0). \quad (18)$$

In Fig. 1 and Fig. 2 are depicted step responses of the system (18) for various values of the parameters.

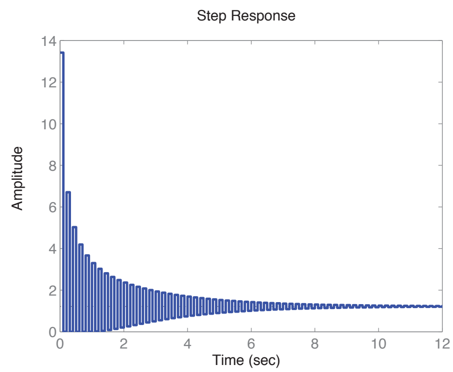

Fig. 1. Step response of system (18) for the parameters $T_d = 3$, $\delta = 0.5$, $\mu = 0$ and $N = 0$ (without filtering derivative action).

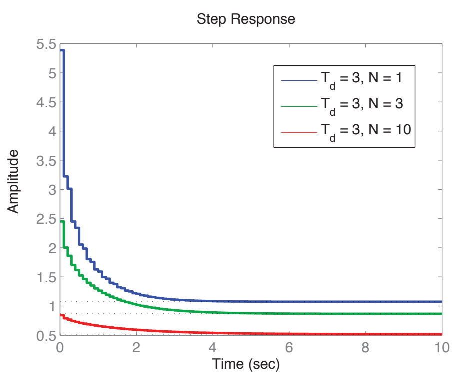

Fig. 2. Step response of system (18) for the parameters $T_d = 3$, $\delta = \mu = 0.5$ and various values of the filter parameter N.

For approximation of the fractional operator s^r, $r \in R$ in results depicted above a CFE method applied on the Tustin rule for sampling period $T = 0.1$ sec and $n = 5$ was used.

VI. Conclusion

This paper has dealt with the design a method for fractional-order controllers implementation, which avoid the effect of actuator saturation and oscillations. By illustrative example has been shown an effect of derivative action filtering. Filtering can be also obtained automatically if the derivative is implemented by taking difference between the reference signal and a its filtered version as it was done in [18]. Instead of filtering just the derivative it is also possible to use usual controller and filter the measured signal. The transfer function of the fractional-order controller with the filter is then

$$C(s) = \frac{U(s)}{E(s)} = \left(K_p + T_i\, s^{-\lambda} + T_d\, s^{\delta}\right) \frac{1}{(T_f s + 1)^{\sigma}}, \quad (19)$$

where σ-th order filter is used. It is an idea for further work.

Acknowledgment

This work was supported in part by the Slovak Grant Agency for Science under grants VEGA: 1/0552/14, 1/0497/11, and by the Slovak Research and Development Agency under the contract No. APVV-0482-11.

References

[1] K. J. Astrom and T. Hagglund, *Advanced PID Control*, USA: ISA, 2006.

[2] R. S. Barbosa, J. A. T. Machado, and M. F. Silva, “Time domain design of fractional differintegrators using least-squares”, Signal Processing, vol. 86, 2006, pp. 2567–2581.

[3] R. Caponetto, G. Dongola, L. Fortuna, and I. Petráš, *Fractional Order Systems: Modeling and Control Applications*, Singapore: World Scientific, 2010.

[4] Y. Q., Chen, “Ubiquitous Fractional Order Controls?”, Proc. of the Second IFAC Symposium on Fractional Derivatives and Applications FDA’06, July 19 - 21, 2006, Porto, Portugal.

[5] Y. Q. Chen, I. Petráš, and D. Xue, “Fractional Order Control - A Tutorial”, Proc. of the American Control Conference, Hyatt Regency Riverfront, St. Louis, MO, USA, June 10-12, 2009.

[6] R. Hilfer, *Applications of Fractional Calculus in Physics*, Singapore: World Scientific Publishers, 2000.

[7] Y. Luo and Y. Q. Chen, *Fractional Order Motion Control*, UK: John Wiley & Sons, 2013.

[8] R. L. Magin, *Fractional Calculus in Bioengineering*, Begell House Publishers, 2006.

[9] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*, Singapore: Imperial College Press, 2010.

[10] C. A. Monje, B. M. Vinagre, V. Feliu, and Y. Q. Chen, “Tuning and auto-tuning of fractional order controllers for industry applications”, Control Engineering Practice, vol. 16, 2008, pp. 798–812.

[11] C. A. Monje, Y. Q. Chen, B. M. Vinagre, D. Xue, and V. Feliu, *Fractional Order Systems and Control - Fundamentals and Applications*, Advanced Industrial Control Series, London: Springer-Verlag, 2010.

[12] K. B. Oldham and J. Spanier, *The Fractional Calculus*, New York: Academic Press, 1974.

[13] A. Oustaloup, *La Derivation Non Entiere: Theorie, Synthese et Applications*, Paris: Hermes, 1995.

[14] F. Padula, A. Visioli, and M. Pagnoni, “On the anti-windup schemes for fractional-order PID controllers”, Proc. of the IEEE 17th Conference on Emerging Technologies & Factory Automation (ETFA), 17-21 September 2012, Krakow, Poland, pp. 1–4.

[15] I. Petráš, “A new discrete approximation of the fractional-order operator”, Proc. of the IEEE ICCC2012, 28-31 May 2012, High Tatras, Slovak Republic, pp. 547–551.

[16] I. Petráš, “An adaptive fractional-order controller, Proc. of the IEEE ICCC2013”, 26-29 May 2013, Rytro, Poland, pp. 297–301.

[17] I. Petráš, “Discrete Fractional-Order PID Controller”, Matlab Central File Exchange, MathWorks, Inc., 2011, url: http://www.mathworks.com/matlabcentral/fileexchange/33761.

[18] I. Petráš, “Practical Aspects of Tuning and Implementation of Fractional-Order Controllers”, Proc. of the ASME 2011-DETC2011, August 28-31, 2011, Washington, USA, pp. 1–10.

[19] I. Petráš, I. Podlubny, P. O’Leary, Ľ. Dorčák, and B. M. Vinagre, *Analogue Realization of Fractional Order Controllers*, FBERG, Technical University of Košice, 2002.

[20] I. Petráš, “Tuning and implementation methods for fractional-order controllers”, Fract. Calc. Appl. Anal., vol. 15, 2012, pp. 282–303.

[21] I. Petráš, *Fractional-Order Nonlinear Systems*, NY: Springer, 2011.

[22] I. Podlubny, *Fractional Differential Equations*, San Diego: Academic Press, 1999.

[23] I. Podlubny, “Fractional-order systems and PI$^{\lambda}$D$^{\mu}$-controllers”, IEEE Transactions on Automatic Control, vol. 44, no. 1, 1999, pp. 208–214.

[24] J. Sabatier, O. P. Agrawal, and J. A. Tenreiro Machado (Eds.), *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, 2007.

[25] B. M. Vinagre, Y. Q. Chen, and I. Petráš, “Two direct Tustin discretization methods for fractional-order differentiator/integrator”, J. Franklin Inst., vol. 340, 2003, pp. 349–362.

[26] B. M. Vinagre, C. A. Monje, A. J. Calderon, and J. I. Suarez, “Fractional PID Controllers for Industry Application. A Brief Introduction”, Journal of Vibration and Control, vol. 13, no. 9-10, 2007, pp. 1419–1429.